Problem 88

Vector $\overrightarrow{\mathbf{B}}$ is 5.0 cm long and vector $\overrightarrow{\mathbf{A}}$ is 4.0 cm long. Find the angle between these two vectors when $\left|\overrightarrow{\mathbf{A}}+\overrightarrow{\mathbf{B}}\right| = 3.0$ cm.

Solution

The length of each vector is given, so the magnitude of each is known.

$$A = \left|\overrightarrow{\mathbf{A}}\right| = 4.0 \text{ cm}$$

$$B = \left|\overrightarrow{\mathbf{B}}\right| = 5.0 \text{ cm}$$

Suppose the two vectors lie in the xy-plane and can be written as $\overrightarrow{\mathbf{A}} = \langle A_x, A_y\rangle$ and $\overrightarrow{\mathbf{B}} = \langle B_x, B_y\rangle$. Let θ be the angle between them. Then

$$\begin{aligned} 3.0 \text{ cm} &= \left|\overrightarrow{\mathbf{A}}+\overrightarrow{\mathbf{B}}\right| \\ &= |\langle A_x, A_y\rangle + \langle B_x, B_y\rangle| \\ &= |\langle A_x + B_x, A_y + B_y\rangle| \\ &= \sqrt{(A_x+B_x)^2 + (A_y+B_y)^2}. \end{aligned}$$

Square both sides.

$$\begin{aligned} 9.0 \text{ cm}^2 &= (A_x+B_x)^2 + (A_y+B_y)^2 \\ &= (A_x^2 + 2A_xB_x + B_x^2) + (A_y^2 + 2A_yB_y + B_y^2) \\ &= (A_x^2 + A_y^2) + (B_x^2 + B_y^2) + 2A_xB_x + 2A_yB_y \\ &= A^2 + B^2 + 2(A_xB_x + A_yB_y) \\ &= A^2 + B^2 + 2\overrightarrow{\mathbf{A}}\cdot\overrightarrow{\mathbf{B}} \\ &= A^2 + B^2 + 2\left|\overrightarrow{\mathbf{A}}\right|\left|\overrightarrow{\mathbf{B}}\right|\cos\theta \\ &= A^2 + B^2 + 2AB\cos\theta \end{aligned}$$

Solve this equation for $\cos\theta$.

$$\cos\theta = \frac{9.0 \text{ cm}^2 - A^2 - B^2}{2AB} = \frac{9.0 \text{ cm}^2 - (4.0 \text{ cm})^2 - (5.0 \text{ cm})^2}{2(4.0 \text{ cm})(5.0 \text{ cm})} = -\frac{4}{5}$$

$$\theta = \cos^{-1}\left(-\frac{4}{5}\right) \approx 143^\circ$$